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LETTER TO THE EDITOR

A group-theoretical derivation of the Hughes–Yadegar algebra

Nigel Backhouse

Department of Applied Mathematics and Theoretical Physics, Liverpool University, PO Box 147, Liverpool L69 3BX, UK

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Abstract. A group-theoretical derivation is given of the Hughes–Yadegar algebra, $O(3)_\Lambda(T_2\times\bar{T}_2)$, and its representations, which shows in particular how the O(4) algebra, the symmetry algebra of the hydrogen atom, arises naturally.

1. Introduction

In a recent paper (Hughes and Yadegar 1976), an analysis was made of the irreducible unitary representations (IUR) of a seven-dimensional algebra, denoted $O(3)_\Lambda(T_2\times\bar{T}_2)$. This algebra was constructed from the O(3) algebra together with two mutually commuting Abelian algebras T_2 and $\bar{T}_2$ which carry the spin-$\frac{1}{2}$ representation of the O(3) algebra. The interesting feature of the algebra is that it possesses IUR which, in a certain sense, contain the bound state hydrogenic IUR of the O(4) algebra.

The purpose of the present letter is threefold: (*a*) to relate the Hughes–Yadegar algebra to the Lie algebra of a Lie group, denoted $SU(2)_\Lambda R^4$; (*b*) by using the method of induced representations to obtain the IUR of the algebra as operators on the space of square-integrable functions on the sphere S^3; (*c*) to explain the success of the algebra in relation to the hydrogen atom by noting how the O(4) algebra naturally fits in with the group scheme.

2. The group $SU(2)_\Lambda R^4$ and its Lie algebra

In order to form the semi-direct product of SU(2) with R^4 we exploit the well known homomorphism of $SU(2)\times SU(2)$ onto SO(4) (see, for example, Talman 1960). We consider the set of all 2×2 complex matrices of the form

$$\mathbf{m}=\begin{pmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{pmatrix},$$

where $\alpha=\tau+i\sigma$, $\beta=\nu+i\mu$ and $(\mu,\nu,\sigma,\tau)\in R^4$. We can check that if $\mathbf{v}\in SU(2)$ then the matrix product $\mathbf{vm}=\mathbf{m}'$ has the same form as $\mathbf{m}$. This gives a linear length-preserving action of SU(2) on the additive group R^4 which embeds SU(2) into SO(4). Now the group $SU(2)_\Lambda R^4$ is the multiplicative group of all 4×4 matrices

$$\begin{pmatrix}\mathbf{v} & \mathbf{m}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}$$

where $\mathbf{v} \in SU(2)$ and $\mathbf{m}$ is a 2×2 matrix of the form defined above. Evidently $SU(2)_\Lambda R^4$ is a subgroup of the Euclidean group in four dimensions.

The Lie algebra of $SU(2)_\Lambda R^4$ is obtained by differentiating the matrices with respect to suitable parameters. We find the following generators:

$$A_1=\begin{pmatrix}-\frac{1}{2}\mathrm{i}\boldsymbol{\sigma}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\qquad A_2=\begin{pmatrix}\frac{1}{2}\mathrm{i}\boldsymbol{\sigma}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\qquad A_3=\begin{pmatrix}\frac{1}{2}\mathrm{i}\boldsymbol{\sigma}_3 & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$$
$$T_\mu=\begin{pmatrix}\mathbf{0} & \mathrm{i}\boldsymbol{\sigma}_1\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\qquad T_\nu=\begin{pmatrix}\mathbf{0} & -\mathrm{i}\boldsymbol{\sigma}_2\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\qquad T_\sigma=\begin{pmatrix}\mathbf{0} & \mathrm{i}\boldsymbol{\sigma}_3\\ \mathbf{0} & \mathbf{0}\end{pmatrix},\qquad T_\tau=\begin{pmatrix}\mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} \tag{1}$$

where

$$\boldsymbol{\sigma}_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \boldsymbol{\sigma}_2=\begin{pmatrix}0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix},\qquad \boldsymbol{\sigma}_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

are the Pauli matrices. Now define the following complex linear combinations of the generators:

$$L_\pm=\pm A_2+\mathrm{i}A_1,\qquad L_0=-\mathrm{i}A_3$$
$$Q_{\frac{1}{2}}=\tfrac{1}{2}(T_\tau-\mathrm{i}T_\sigma)\qquad Q_{-\frac{1}{2}}=\tfrac{1}{2}(T_\nu-\mathrm{i}T_\mu)$$
$$\bar{Q}_{\frac{1}{2}}=-\tfrac{1}{2}(T_\nu+\mathrm{i}T_\mu)\qquad \bar{Q}_{-\frac{1}{2}}=\tfrac{1}{2}(T_\tau+\mathrm{i}T_\sigma).$$

We can write these quite explicitly in terms of the basic 4×4 unit matrices: if E_{rs} is the 4×4 matrix with one in the (rs) position and zero elsewhere, for $r, s = 1, 2, 3, 4$, then we find $L_+=E_{12}$, $L_-=E_{21}$, $L_0=\frac{1}{2}(E_{11}-E_{22})$, $Q_{\frac{1}{2}}=E_{13}$, $Q_{-\frac{1}{2}}=E_{23}$, $\bar{Q}_{\frac{1}{2}}=E_{14}$, $\bar{Q}_{-\frac{1}{2}}=E_{24}$. It is clear that the Q commute. The other commutation rules are:

$$[L_0, L_\pm]=\pm L_\pm,\qquad [L_+, L_-]=2L_0$$
$$[L_0, \overset{(-)}{Q}_{\pm\frac{1}{2}}]=\pm\tfrac{1}{2}\overset{(-)}{Q}_{\pm\frac{1}{2}},\qquad [L_\pm, \overset{(-)}{Q}_{\mp\frac{1}{2}}]=\overset{(-)}{Q}_{\pm\frac{1}{2}}, \tag{2}$$

which are precisely the commutation rules for the Hughes–Yadegar algebra. So the latter appears as a real subalgebra of the complexification of the Lie algebra of the Lie group $SU(2)_\Lambda R^4$.

3. IUR of $SU(2)_\Lambda R^4$

In Hughes and Yadegar (1976) the IUR of $O(3)_\Lambda(T_2\times \bar{T}_2)$ were computed using shift operator techniques. We use another method here, namely the theory of group representations as applied to semi-direct product groups. First note that a unitary representation U of the group $SU(2)_\Lambda R^4$ gives rise to a representation of the infinitesimal generators by skew-Hermitian operators. It quickly follows that

$$U(L_0)^\dagger=U(L_0),\qquad U(L_\pm)^\dagger=U(L_\mp),\qquad U(Q_{\pm\frac{1}{2}})^\dagger=\mp U(\bar{Q}_{\mp\frac{1}{2}}). \tag{3}$$

These Hermiticity conditions differ slightly from those given in equation (2) of Hughes and Yadegar (1976). However, Hughes (1976, private communication) informs me that the above conditions (3) are now favoured. This has the consequence that the Casimir invariant $X=Q_{\frac{1}{2}}\bar{Q}_{-\frac{1}{2}}-Q_{-\frac{1}{2}}\bar{Q}_{\frac{1}{2}}$ is a negative definite Hermitian operator in the representation.

The IUR of $SU(2)_\Lambda R^4$ can be determined using Mackey's method of induced representations for regular semi-direct product groups based on the linear characters of the normal Abelian subgroup R^4 (see Mackey 1968). Denote by χ^{k}, $\boldsymbol{k} = (k_1, k_2, k_3, k_4)$, the linear character of R^4 whose value at $\mathbf{m} \in R^4$ is $\exp[i(k_1\mu + k_2\nu + k_3\sigma + k_4\tau)]$. It can be checked that the action of SU(2) on the dual space $\hat{R}_4$ (the space of linear characters) is transitive on the spheres $\boldsymbol{k}\cdot\boldsymbol{k} =$ constant. It is convenient to take the point $(\lambda, 0, 0, 0)$ as the representative point on the orbit $\boldsymbol{k}\cdot\boldsymbol{k} = \lambda^2$. Then the little co-group associated with this point is SU(2) if $\lambda = 0$, but trivial otherwise. An IUR of $SU(2)_\wedge R^4$ which corresponds to $\lambda = 0$ is composed of an IUR of SU(2) and the trivial representation of R^4. X is the zero operator in this representation. The Mackey theory tells us that the induced representations $U^\lambda = \chi^{(\lambda,0,0,0)} \uparrow SU(2)_\wedge R^4$ are irreducible for $\lambda \neq 0$ and, furthermore, we obtain a representative from each of the remaining equivalence classes of IUR by considering all $\lambda > 0$.

The induced representation U^λ is defined on the space of all complex-valued square-integrable functions on $(SU(2)_\wedge R^4)/R^4 \cong SU(2)$. The action is as follows: let $f \in L^2(SU(2))$ then

$$\begin{pmatrix} \mathbf{v} & \mathbf{m} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \in SU(2)_\wedge R^4$$

maps f to f' where $f'(\mathbf{w}) = \chi^{(\lambda,0,0,0)}(\mathbf{wm})f(\mathbf{wv})$ for all $\mathbf{w} \in SU(2)$. In particular, putting $\mathbf{m} = \mathbf{0}$, we get the regular representation of SU(2). To get a more concrete realization we first note that the group manifold SU(2) is homeomorphic to the sphere S^3. Using the coordinates (μ, ν, σ, τ) for the four-space in which the sphere $\mu^2 + \nu^2 + \sigma^2 + \tau^2 = 1$ is embedded we find that the Q are represented by the multiplication operators

$$\begin{aligned} U^\lambda(Q_{\frac12}) &= \tfrac12\lambda(\nu + i\mu), & U^\lambda(Q_{-\frac12}) &= \tfrac12\lambda(\tau - i\sigma), \\ U^\lambda(\bar{Q}_{\frac12}) &= \tfrac12\lambda(\tau + i\sigma), & U^\lambda(\bar{Q}_{-\frac12}) &= \tfrac12\lambda(-\nu + i\mu). \end{aligned} \tag{4}$$

The Casimir invariant X assumes the value $-\lambda^2/4$ on $L^2(S^3)$. The SU(2) generators are represented by the following differential operators:

$$\begin{aligned} U^\lambda(A_1) &= \frac12\left(\mu\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\mu} + \nu\frac{\partial}{\partial\sigma} - \sigma\frac{\partial}{\partial\nu}\right), \\ U^\lambda(A_2) &= \frac12\left(\nu\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\nu} + \sigma\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\sigma}\right), \\ U^\lambda(A_3) &= \frac12\left(\tau\frac{\partial}{\partial\sigma} - \sigma\frac{\partial}{\partial\tau} + \nu\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\nu}\right). \end{aligned} \tag{5}$$

Note that these operators are independent of λ.

In the definition of U^λ we used a right action of SU(2) on $L^2(S^3)$. There is also a natural left action given by $\mathbf{v}: f \to f'$ where $f'(\mathbf{w}) = f(\mathbf{v}^{-1}\mathbf{w})$ for $\mathbf{v}, \mathbf{w} \in SU(2)$. If we denote the generators of this new SU(2) group by A'_1, A'_2, A'_3 then in the representation we find

$$\begin{aligned} U^\lambda(A'_1) &= \frac12\left(\tau\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\tau} + \nu\frac{\partial}{\partial\sigma} - \sigma\frac{\partial}{\partial\nu}\right) \\ U^\lambda(A'_2) &= \frac12\left(\tau\frac{\partial}{\partial\nu} - \nu\frac{\partial}{\partial\tau} + \sigma\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\sigma}\right) \\ U^\lambda(A'_3) &= \frac12\left(\sigma\frac{\partial}{\partial\tau} - \tau\frac{\partial}{\partial\sigma} + \nu\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\nu}\right). \end{aligned} \tag{6}$$

These differential operators can be derived in another way. In fact if we evaluate the operator $\hat{Y}_0 = Y_0/X$ (see (6) of Hughes and Yadegar 1976) using (4), (5) above, we get the same expression as $U^\lambda(A_3')$. The other expressions are obtained similarly. Thus the fact that Hughes and Yadegar were able to find another O(3) algebra in the universal enveloping algebra of $O(3)_\Lambda(T_2 \times \bar{T}_2)$ is just a reflection of the existence of the second action of SU(2). Together, the left and right actions of SU(2) give a representation of $SU(2) \times SU(2)$, the covering group of SO(4), on $L^2(S^3)$.

We have remarked that the restriction to SU(2) of U^λ is the regular representation. We can also restrict U^λ to the diagonal subgroup of $SU(2) \times SU(2)$—this in fact gives a single-valued representation of SO(3). It is well known that this representation is the hydrogenic representation $\oplus(D^j \otimes D^j)$, summed over all IUR D^J of SU(2) for $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (see Backhouse and Gard 1976). The infinitesimal generators of this action are

$$
\begin{aligned}
U^\lambda(A_1) + U^\lambda(A_1') &= \nu\frac{\partial}{\partial\sigma} - \sigma\frac{\partial}{\partial\nu}, \\
U^\lambda(A_2) + U^\lambda(A_2') &= \sigma\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\sigma}, \qquad (7) \\
U^\lambda(A_3) + U^\lambda(A_3') &= \nu\frac{\partial}{\partial\mu} - \mu\frac{\partial}{\partial\nu}.
\end{aligned}
$$

These resemble the angular momentum operators in three-space. The complementary operators do not close under commutation, so do not generate a group. Indeed, they obey the same commutation rules as the components of the Runge–Lenz vector, which arises in the context of the hydrogen atom problem.

4. Conclusions

We have written down the group behind the algebra $O(3)_\Lambda(T_2 \times \bar{T}_2)$ and shown how its IUR can be obtained. We recall from the work of Fock, as expounded and expanded by Bander and Itzykson (1966), that the accidental degeneracy of the hydrogen atom spectrum can be explained by first transforming into three-momentum space and then stereographically projecting onto the O(4) invariant sphere S^3 in four-space. Our approach reconstructs this sphere with a natural group action and so explains the success of the Hughes–Yadegar algebra in relation to the hydrogenic bound state wavefunctions.

Our differential operator expressions for the actions of the generators on $L^2(S^3)$ are currently being investigated by Hughes (1976, private communication) as a tool for the evaluation of matrix elements, between hydrogenic wavefunctions, of certain operators—in particular the momentum operators.

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